The Classification of Magnetic Shells

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ABSTRACT

A new method of calculating the magnetic shell parameter L, based on Pennington's perturbation method for calculating the shape of magnetic shells, is presented. In this approach, the shell passing a given point is characterized by two parameters, by L and by a function θ_m of the local pitch angle. However, the dependence on θ_m is shown to be very weak, so that it may be neglected to a good approximation. Auxilliary functions necessary for this method are tabulated, including a 48-coefficient expansion of the geomagnetic potential in tilted-dipole coordinates, and the results are compared to those derived by McIlwain's method. The results are found to be, in general, within several percent of those numerically integrated by McIlwain.

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Introduction

This work deals with the motion of charged particles trapped in the geomagnetic field and of low enough energy for the guiding center approximation to hold. To the lowest order of approximation each of these particles will be tied to a line of force, oscillating back and forth along it and getting reflected from points at which the field intensity B reaches a "mirroring value" B_m, depending on the particle's magnetic moment. In the next order of approximation a slow drift from one line of force to another is added, the particle drifting (on the average) to that one of all adjacent lines of force on which the longitudinal invariant

$$I = \int v_{\parallel}\, dl$$

between points with intensity B_m maintains its value. In general, this drift motion will cause the guiding center to follow a surface termed a magnetic shell. A group of trapped particles with the same B_m and I will share the same shell at all times.

Consider a magnetic field defined in the space outside a sphere of radius a and deviating only slightly, at any point in that space, from the field of a magnetic dipole at the sphere's center. On a cross-cut of the field (which can be regarded as a generalization of the meridional plane - a surface everywhere tangent to $\underline{B}$ and bounded by that line of force which extends to infinity) there will

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in general be one and only one line of force on which I corresponds to a given B_m; the shell corresponding to any (B_m, I) is then unique. Any given shell will, of course, share its line of force with many others but, unless the field is axisymmetric, on different cross-cuts it will in general share lines of force with different shells. In that case, then, magnetic shells need two parameters, such as B_m and I, for their classification.

In an axisymmetric field all meridional cross-cuts have the same appearance and magnetic shell surfaces may be described by a single parameter; such shells have been termed degenerate by Stone (1963). For instance, each of the surfaces

$$r = r_o \sin^2\theta \qquad (1)$$

formed in a dipole field by the rotation of a line of force around the symmetry axis, contains an entire family of magnetic shells, each with a different I and a different B_m or, equivalently, different I and reflection colatitude θ_m. In analyzing the motion of charged particles in such a field it is much preferable to label shells by r_o and B_m, first because the family of particles with the same r_o occupies the same region in space and secondly because scattering of such particles leaves r_o almost unchanged, thus affecting mainly the distribution of particles within the family.

The geomagnetic field is not a dipole field, but it approximates one closely enough to warrant treating the shells as degenerate. In a perturbed dipole field the approximate shape of shells may be calculated (Pennington, 1961) by regarding them as perturbations of dipole shells given by eq. (1). Given a magnetic shell in a perturbed dipole field, we'll call that dipole shell of which it is a perturbation its <u>ancestor</u> and the distance r_o which characterizes the dipole shell by equation (1) its <u>ancestry parameter</u> or "ancestry" in short. Because the perturbed shells are no longer degenerate, the family of shells obtained from a given dipole shell will depend not only on their ancestry r_o but also upon their reflection point B_m; however, for moderate perturbations this dependence is expected to be weak and perturbed shells of the same ancestry lie close together.

So far ancestry has not been uniquely defined: if in a perturbed dipole field the shape of a given magnetic shell deviates, on the average, by an amount δ from that obtained when nondipole components are neglected, there exists a whole range of dipole shells, with thickness of order δ, any of which might be considered as its "ancestor". There are many possible ways of defining ancestry. For instance, one could start with a dipole field in which charged particles are trapped, apply the perturbation gradually and define as the ancestor of a given shell that dipole shell from which its particles originated. In practice, this would involve the third adiabatic invariant and is far too complicated. Stone (1963) has proposed using as ancestry parameter for particles on a given line of force either the maximum distance R_o of the line from the origin or the parameter r_o of that dipole line having the same minimum field intensity B_o. Unfortunately, neither R_o nor B_o are constant on a given shell (though Stone shows their variation to be relatively small); to obtain a unique ancestry definition in this manner, one has to average the ancestry parameter over the entire shell rather than use its value at a random line of force on the shell, which may vary from one line to another. However, the most widely used method of defining ancestry and probably the best one is due to McIlwain (1961) who defines r_o as belonging to that dipole shell on which B_m corresponds to the same I. It has the great advantage that in order to identify a shell, it requires integration along one of its lines of force only with no reference to the rest of the shell. The ancestry parameter thus defined is usually denoted by L; to derive it, I is numerically evaluated, entailing a relatively lengthy calculation, after which an empirical relationship due to McIlwain (1961) is used.

Pennington's approximate derivation of the shell equations (Pennington, unpublished) defines ancestry in the same manner as McIlwain's. These shell equations (Pennington, 1961) are of the form

$$r = L \sin^2\theta + \sin^2\theta \, R_1 \, (L, \theta_m, \theta, \varphi) \tag{2}$$

Here θ_m is the reflection colatitude, in the dipole field obtained when higher harmonics are neglected and on the shell with $r_o = L$, of particles with the given B_m

$$B_m = g_1^0 \, (r_0 \sin^2\vartheta)^{-3} \, (1 + 3\cos^2\vartheta)^{1/2} \tag{3}$$

and R_1 is relatively small and is a linear function of the higher harmonics g_n^m and h_n^m of the geomagnetic potential. Obviously, a first order approximation of L is

$$L = r/\sin^2\vartheta - R_1\,(r/\sin^2\vartheta, \vartheta_m, \vartheta, \varphi) \tag{4}$$

This definition of L has the advantage of explicit dependence on θ_m (though, as expected, the dependence is rather weak) and of simplicity - the numerical integration of I is not required and instead, a simple substitution formula is used. It suffers, as any first-order perturbation method, from the neglection of second-order terms, especially near the earth's surface. To reduce this source of error, it is advantageous (as pointed out by Pennington) to start from the tilted-dipole or the eccentric-dipole approximation to the geomagnetic field.

Calculation

The Geomagnetic Potential

In order to reduce the second-order error, the 48-term expansion of the geomagnetic potential was transformed into tilted dipole coordinates. This was accomplished by a "brute force" technique: using a given expansion, the potential was calculated at 48 selected points which were then transformed into tilted coordinates, yielding a set of 48 linear equations in the 48 new coefficients. The linear equations were solved using an IBM 7094 computer. As an accuracy check, the calculation was repeated with a different set of reference points; the difference between the two sets thus obtained was in the fifth significant figure.

The set of coefficients used consists of the first 48 terms in the expansion by Cain, Daniels, Hendricks and Jensen (unpublished) for the epoch 1960. The pole of the tilted dipole coordinate system is then

colatitude $\theta_0 = 11^{\circ}41'$

longitude $\theta_0 = -68^{\circ}57'$

and the field expansion coefficients in tilted dipole coordinates are given in Table 1. The method is not suitable for the eccentric dipole approximation: a 48-term potential given in geographical coordinates needs an infinite number of terms for its description in eccentric dipole coordinates and a truncation at 48 terms leads in that case to appreciable error.

The Equation for L

In tilted dipole coordinates Pennington's function R_1, appearing in equation (2), is (Pennington, 1961).

$$R_1 = \frac{a}{g_1^0} \sum_{n=2} \sum_{m=0}^{n} \left(\frac{a}{L}\right)^{n-2} \left[V_n^m(\vartheta) - \alpha_n^m(\vartheta_m) \right] \left[g_n^m \cos m\varphi + h_n^m \sin m\varphi \right] \qquad (5)$$

where a is the earth's radius and the g_n^m and h_n^m are harmonic coefficients of the geomagnetic potential. The V_n^m are trigonometric integrals which can be evaluated analytically while the α_n^m have to be integrated numerically and vanish for even n + m. In the equatorial plane the nonvanishing α_n^m tend to a limiting value

$$\alpha_n^m(\pi/2) = -\frac{1}{3}\frac{dP_n^m}{d\vartheta}(\pi/2)$$

where $P_n^m(\vartheta)$ is an associated Legendre polynomial.

The coefficients of the V_n^m and numerical values of the α_n^m, up to n = 6, are given in tables 2, 3 and 4; computer programs which will derive them for any reasonable n and m are available upon request. The numerical integration of α_n^m involves an integrand which diverges at the mirror point; it therefore has to be done with considerable precision, especially near the mirror point. The error in the values of α_n^m derived here has been estimated and is typically in the 5-th decimal digit. Thus for n less than 4 they replace these previously published (Pennington, 1961).

It will be noted that while the shells of an axisymmetrically perturbed dipole are degenerate, they may appear to be nondegenerate when this method is used, due to the non-vanishing α_n^o. One can redefine ancestry so that this no longer occurs, though the improvement is of little practical importance. For the new definition, let I' be the longitudinal invariant of a given shell after the contributions from axisymmetric harmonics have been subtracted. The ancestry parameter L of the shell is then r_0 of the dipole (B_m, I') shell obtained when all higher harmonics are neglected. In practice this redefinition amounts to setting all α_n^o equal to zero. Since the departure from degeneracy is small in the first place and the α_n^o contribute only a small part of it, this is of little practical consequence.

Results and Conclusions

The perturbation method was tested by comparison with INVAR, a computer program for calculating L due to McIlwain. As a sample result, values obtained at the earth's surface ($r = a$) and along northern latitude 60° are given in table 5. As can be seen, the discrepancy reaches 6%, which at $L = 3$ corresponds to a displacement of about 1°. A similar scan along the equator also gives deviations of up to 6% which, in that case, can correspond to quite large displacements; it should be noted, however, that most of this is due to a systematic difference between the two methods. This was shown by calculating L by the perturbation method for a series of near-equatorial pairs of conjugate points: only rarely did the difference of L for the pair exceed 1% and it did so only for relatively large values of L, so that the corresponding displacements were within 2°. All preceding results were derived for points on the earth's surface; at larger distances the perturbation method is expected to improve progressively.

The ϑ_m dependence of L is given in Table 6, listing L for particles with pitch angles between 15° and 90° in the equatorial plane of the tilted dipole, for various tilted-dipole longitudes and for radial distances of 1.5 and 4 earth raddi (the choice of points does not entail equal L though the variation is relatively small). As can be seen, for any point L depends only weakly on ϑ_m.

The preceding results clearly demonstrate the feasibility of Pennington's perturbation method for deriving L when accuracy is not critical, especially for higher latitudes. A computer program based on this method, and using the functions tabulated here required 16 seconds to be set up initially and thereafter 50msec for each calculation of L (assuming the particles mirror at the given point, i.e., $\vartheta_m = \vartheta$) while INVAR, on the same computing system, required several seconds, depending on latitude. The perturbation method does not calculate B or I, but tilted dipole coordinates of the given point are easily obtained as a by-product.

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Captions for Tables

Table 1: Schmidt normalized harmonic expansion coefficients of the geomagnetic potential in tilted dipole coordinates. Given in units of 10^{-4} gauss and derived from the epoch 1960 expansion of Cain, Daniels, Hendricks and Jensen.

Table 2: The coefficients of the trigonometric integrals $V_n^m(\theta)$ for odd m ~~even m~~.

Table 3: The coefficients of the trigonometric integrals $V_n^m(\theta)$ for even m ~~odd m~~.

Table 4: The functions α_n^m against the mirroring colatitude θ_m.

Table 5: Comparison of magnetic shell parameters L_1 obtained by McIlwain's program and L_2 obtained by the perturbation method, for various longitudes along northern latitude 60°.

Table 6: The magnetic shell parameter L as function of pitch angle λ, obtained by the perturbation method in the dipole equator, for various dipole longitudes, at distances of 1.5 at 4 earth radii.

References

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Table 1

n	m	0	1	2	3	4	5	6
1	g	-31408						
2	g	- 520	2916	-1903				
	h		2255	444				
3	g	936	-1211	-1034	-561			
	h		-1740	1166	-541			
4	g	838	- 504	- 11	326	-322		
	h		961	195	- 39	-323		
5	g	-144	117	-311	65	-137	-21	
	h		328	- 60	157	99	56	
6	g	45	- 28	74	194	-150	13	-60
	h		46	- 33	85	- 41	-47	-68

Table 2

V_n^m for odd m

n	m	$\sin^{-1}\theta$	$\sin^{-3}\theta$	$\sin^{-5}\theta$	$\sin^{-7}\theta$	$\sin^{-9}\theta$	$\sin^{-11}\theta$	1
n+m odd								
n+m even		multiply by cos θ						log. tan(θ/2)
1	1	-2.0000						
2	1	1.7321	-1.1547					-.5774
3	1	.2449	3.1843	- .9798				
3	3	-1.5811	-1.5811					
4	1	-5.5340	6.0083	- .9035				.4292
4	3		2.0917	- 2.5100				.4183
5	1	- .1598	- .0799	-10.2265	8.4222	- .8607		
5	3	1.0757	.5379	5.1096	- 3.5857			
5	5	-1.8708	- .9354	- 1.4031				
6	1			18.9031	-29.9504	12.2202	-.8332	-.3397
6	3			- 9.9628	15.2677	- 4.8305		-.4744
6	5			2.3268	- 3.3240			.9972

Table 3

V_n^m for even m

n	m	$\sin^{-2}\theta$	$\sin^{-4}\theta$	$\sin^{-6}\theta$	$\sin^{-8}\theta$	$\sin^{-10}\theta$	1
n+m odd							
n+m even		multiply by cos θ					log tan(θ/2)
2	0	1.5000					
2	2	-1.7321					.8660
3	0	-2.5000	2.0000				.5000
3	2	1.9365	-1.9365				
4	0	~~.0000~~	-4.3750	2.5000			~~.0000~~
4	2	.4193	4.1926	-2.2360			- .4193
4	4	-1.1093	-1.4790				1.1093
5	0		7.8750	-10.5000	3.0000		- .3750
5	2		-7.6852	10.2470	- 2.5617		~~.0000~~
5	4		2.2185	- 2.9580			.7395
6	0	~~.0000~~	~~.0000~~	14.4375	-15.7500	3.5000	~~.0000~~
6	2	- .3396	- .2264	-15.1254	15.7594	-2.8983	.3396
6	4	.9301	.6201	5.9529	- 4.9608		- .9301
6	6	-1.2594	- .8396	- 1.3434			1.2594

Table 4

θ	α_2^1	α_3^0	α_3^2	α_4^1	α_4^3	α_5^0	α_5^2	α_5^4	α_6^1	α_6^3	α_6^5
10	-0.645	-9.220	-3.578	-394.02	-20.909	-9814.5	-5398.5	-134.39	~~-573449.~~	-56959.	-894.22
15	-0.600	-3.336	-2.436	-77.956	-9.011	-702.12	-626.28	-34.649	-21963.	-4011.6	-139.05
20	-0.587	-1.267	-1.995	-26.506	-5.831	-63.571	-160.52	-16.819	-2155.4	-760.23	-49.384
25	-0.582	-0.366	-1.723	-10.811	-4.304	2.709	-56.017	-10.292	-292.63	-217.93	-24.502
30	-0.579	0.074	-1.523	-4.611	-3.372	7.264	-22.524	-7.031	-35.498	-76.927	-14.367
35	-0.578	0.303	-1.363	-1.811	-2.732	4.886	-9.555	-5.120	3.073	-30.314	-9.308
40	-0.577	0.425	-1.230	-0.447	-2.254	2.648	-3.933	-3.888	6.384	-12.462	-6.442
45	-0.577	0.489	-1.118	0.243	-1.906	1.201	-1.324	-3.148	4.174	-4.947	-4.670
50	-0.577	0.521	-1.022	0.593	-1.625	0.346	-0.071	-2.429	2.107	-1.597	-3.501
55	-0.577	0.533	-0.939	0.764	-1.398	-0.140	0.530	-1.972	0.721	-0.066	-2.689
60	-0.576	0.535	-0.867	0.838	-1.213	-0.405	0.805	-1.621	-0.104	0.621	-2.102
70	-0.576	0.522	-0.751	0.852	-0.934	-0.606	0.935	-1.128	-0.799	0.991	-1.336
80	-0.577	0.506	-0.673	0.812	-0.759	-0.631	0.888	-0.839	-0.945	0.952	-0.916
90	-0.577	0.500	-0.645	0.791	-0.697	-0.625	0.854	-0.740	-0.955	0.906	-0.776

Table 5

φ	L_1	L_2
0	3.7195	3.8595
10	3.4802	3.6642
20	3.3200	3.5079
30	3.2162	3.3743
40	3.1499	3.2558
50	3.1050	3.1490
60	3.0678	3.0548
70	3.0314	2.9715
80	2.9924	2.8997
90	2.9509	2.8391
100	2.9067	2.7910
110	2.8602	2.7527
120	2.8124	2.7249
130	2.7685	2.7081
140	2.7358	2.7065
150	2.7258	2.7249
160	2.7525	2.7717
170	2.8241	2.8564

φ	L_1	L_2
- 10	4.0634	4.1293
- 20	4.5543	4.5433
- 30	5.2430	5.2157
- 40	6.1877	6.2795
- 50	7.4225	7.7750
- 60	8.8611	9.4537
- 70	10.1520	10.7106
- 80	10.7049	10.9890
- 90	10.1781	10.2327
-100	8.8848	8.8774
-110	7.4136	7.4305
-120	6.1332	6.1634
-130	5.1315	5.1659
-140	4.3836	4.4182
-150	3.8366	3.8704
-160	3.4367	3.4740
-170	3.1497	3.1904
-180	2.9533	2.9909

Table 6

φ / λ	$r/a = 1.5$			$r/a = 4$		
	0	100	200	0	100	200
90	1.5383	1.5163	1.4243	4.0463	4.0295	3.9304
70	1.5379	1.5161	1.4243	4.0461	4.0295	3.9304
50	1.5367	1.5163	1.4240	4.0455	4.0298	3.9302
40	1.5357	1.5173	1.4236	4.0449	4.0300	3.9301
30	1.5343	1.5204	1.4227	4.0442	4.0307	3.9300
25	1.5333	1.5240	1.4216	4.0437	4.0313	3.9299
20	1.5322	1.5309	1.4193	4.0431	4.0324	3.9299
15	1.5307	1.5454	1.4138	4.0423	4.0345	3.9300